Topology and its Applications 34 (1990) 69-91
North-Holland

THE MAXIMAL TOTALLY BOUNDED GROUP TOPOLOGY ON G AND THE BIGGEST MINIMAL G-SPACE, FOR ABELIAN GROUPS G

Eric K. van DOUWEN*

For an Abelian (abstract) group G let bG denote the Bohr-compactification of G and let G^* denote G as topological subgroup of bG, or, equivalently, let G^* be G with its maximal totally bounded group topology. Our main result says that G^* has "many" discrete C-embedded subspaces which are C^*-embedded in bG. A consequence is that no sequence in G^* converges to a point of bG.

As an application we get information about BG, the unique (up to isomorphism) biggest G-space, for (abstract) Abelian G. We show that $\pi(BG) > |G|$ and $|BG| = \exp^2|G|$. For countable Abelian G this tells us BG is not the absolute of ${}^{\omega}2$.

AMS (MOS) Subj. Class.: 54B99, 54C20, 54G99, 22A99, 54H20, 20K99

maximal totally bounded topology	Abelian group
biggest minimal G-space	G-space

1. Introduction

The motivation for this paper is the question of what BG, the biggest minimal G-space, is for abstract groups G, see Section 1.2 below. For Abelian G our strategy for studying BG is to use the Bohr-compactification bG of G: It is a more accessible G-space, and we can transfer some information from bG to BG. Note that this is a rather weak strategy, since it does not really exploit the fact that BG is the biggest minimal G-space. Also, we can say something about this auxilliary space only if G is Abelian.

Most of this paper deals with bG, and in fact with $G^{\#}$, i.e., with G as topological subgroup of bG. The remainder of this introduction will reflect this shift of emphasis: We first discuss $G^{\#}$ and bG, and then BG.

Before we really start our introduction we briefly consider ways a subspace can be nice: As usual, call a function a *map* if it is continuous. Also, let $\mathbb{I}$, $\mathbb{N}$, $\mathbb{R}$ and 2 denote the unit interval [0, 1], the positive integers, the reals, and {0, 1} respectively. If S is a space we say that a subspace A of a space X is *S-embedded* if every map

* Partially supported by an NSF Grant.

0166-8641/90/$3.50 © 1990, Elsevier Science Publishers B.V. (North-Holland)

$A \to S$ extends to a map $X \to S$. So

$$\mathbb{R}\text{-embedded} \equiv C\text{-embedded}, \qquad \mathbb{I}\text{-embedded} \equiv C^*\text{-embedded}.$$

We mention that if A is strongly zero-dimensional, in particular, if A is discrete, then (see Lemma 2.1.1)

$$\begin{array}{ccc} A \text{ is } \mathbb{N}\text{-embedded} & \Rightarrow & A \text{ is } 2\text{-embedded} \\ \Downarrow & & \Downarrow \\ A \text{ is } \mathbb{R}\text{-embedded} & \Rightarrow & A \text{ is } \mathbb{I}\text{-embedded}. \end{array}$$

We also remind the reader that a discrete space is real-compact iff its cardinality is not Ulam-measurable. Therefore, if D is a relatively discrete $\mathbb{R}$-embedded subspace of a space[1] X and if the cardinality of D is not Ulam-measurable, then D is closed in X.

1.1. G^* and bG

For the purpose of this introduction it suffices to know that the *Bohr-compactification bG* of an abstract Abelian group G is the biggest compact topological group which algebraically has G as a dense subgroup. We use G^* to denote G as a topological subgroup of bG; an internal description of the topology of G^* is that it is the biggest totally bounded group topology on G.

G^* has been considered before [1, 2]. We discuss its basic properties, including the little that is known already, in Section 4. One of the new basic results we obtain is the following, proved as Theorem 4.8:

1.1.1. Theorem. *G^* is zero-dimensional, for every Abelian group G.*

If $\mathscr{A}$ is a collection of sets all of cardinality κ for some cardinal κ, then we call D a *transversal* for $\mathscr{A}$ if $(\forall A \in \mathscr{A})[|A \cap D| = \kappa]$.

The main result of this paper is:

1.1.2. Lemma. *Let G be an infinite Abelian group, and let κ be a cardinal with $\omega \leqslant \kappa \leqslant |G|$. Let $\mathscr{A}$ be a collection of at most κ subsets of G, each of cardinality exactly κ. Then $\mathscr{A}$ has a transversal D such that:*

(a) *D is relatively discrete (in G^*, or, equivalently, in bG),*

(b) *D is $\mathbb{N}$-embedded in G^*; and*

(c) *D is $\mathbb{I}$-embedded in bG.*

The motivation for this result is that it answers a question about BG, see Section 1.2. It also is useful for the study of G^*: An obvious corollary is the following:

1.1.3. Theorem. *If G is an Abelian group, then*

(a) *every infinite subset A of G^* has a relatively discrete subset D with $|D| = |A|$ that is $\mathbb{N}$-embedded in G^* and is $\mathbb{I}$-embedded in bG; hence*

[1] All spaces are assumed to be Tychonoff spaces, unless the contrary is clear from the context.

(b) *for all infinite* $A \subseteq G$ *and* $p \in bG$ *there is a neighborhood* U *of* p *in* bG *such that* $|A \setminus U| = |A|$ *(i.e., "S does not converge to p"); in particular*

(c) *no nontrivial sequence in* $G^{\#}$ *converges to a point of* bG.

Theorem 1.1.3(a) strengthens the fact that if G is an infinite Abelian group, then $G^{\#}$ is not pseudocompact [2, 2.2]. See Corollary 4.7 for another strengthening. Part (a) also implies that G has a closed discrete subset of cardinality κ for every $\kappa \leqslant |G|$ that is not Ulam-measurable. We do not know if the restriction on κ is essential. We also do not know if G has a relatively discrete subset that is not closed.

Part (c) is of interest by itself; it should be compared with the fact that every infinite compact group has a nontrivial convergent sequence.[2] It also has an application dealing with products: If G is the product of an indexed collection $\langle G_i : i \in I \rangle$ of nondegenerate Abelian groups, then $G^{\#} = \prod_{i \in I} (G_i^{\#})$ iff I is finite, [2, 2.5 and 2.6]. The "if" part will be reproved as Fact 4.5; using part (c) we can give a purely topological explanation of the "only if" part: $G^{\#}$ has no nontrivial convergent sequences, but of course $\prod_{i \in I} G_i^{\#}$ has nontrivial convergent sequences if I is infinite.

Does theorem 1.1.3 generalize? If we weaken "abstract", i.e., discrete, to "locally compact", then there still is a "largest" compact topological group bG such that the underlying group of G is a dense subgroup of bG, call it $G^{\#}$, and such that id_G is continuous as function from G to $G^{\#}$. For $G = \mathbb{R}$ we can generalize the case $|A| = \omega$ of Theorem 1.1.3(b):

1.1.4. Theorem. *For a subset* A *of* $\mathbb{R}$ *the following are equivalent*:

(a) $\mathrm{cl}_{\mathbb{R}} A$ *is not compact*;

(b) *A has an infinite subset D that is relatively discrete in* $\mathbb{R}^{\#}$ *and that is* $\mathbb{I}$*-embedded in* $b\mathbb{R}$;

(c) *A has an infinite subset D that is closed discrete in* $\mathbb{R}^{\#}$ *and that is* $\mathbb{R}$*-embedded in* $\mathbb{R}^{\#}$ *and is* $\mathbb{I}$*-embedded in* $b\mathbb{R}$.

We do not know whether an analogous result holds for all locally compact Abelian groups.

1.2. G-spaces

We call an *action* of a group G on a space X any homomorphism, h say, from G into the autohomeomorphism group of X We call X a *G-space* if G acts on X; the action of G on X, h say, will always be clear from the context, and we will write gx for $h(g)(x)$, $x \in X$, $g \in G$.[3] The *orbit* of $x \in X$ is $Gx = \{gx : g \in G\}$.

[2] See [10, 2.5.2] for an easy proof for the special case of compact Abelian groups. Infinite compact groups have nontrivial convergent sequences since they are dyadic, [9], see also [11].

[3] Our concept of action coincides with the usual concept, but is simpler. (It can be simpler since we do not deal with continuous actions.)

If G acts on X, a subset A of X is called *invariant* if $A \neq \emptyset$ and if $(\forall g \in G)$ $[gA = A]$;[4] in that case A is a G-space, with obvious action.

One calls a subset M of a G-space X a *minimal* subset of X if M is a minimal (under inclusion) member of the collection of nonempty closed invariant subsets of X, or, equivalently, if each point of M has its orbit dense in M. One calls a G-space X *minimal* if X is a minimal subset of itself. Clearly, every minimal subset of a G-space is itself a minimal G-space. As every compact G-space has a minimal subset, because of Zorn's Lemma [5, 2.4], it follows that there is an abundance of compact minimal G-spaces.

If X and Y are G-spaces, we call a G-map any map $f: X \to Y$ which commutes with the action of G, i.e.,

$$(\forall x \in X)(\forall g \in G)[f(gx) = gf(x)].$$

Note that the range of a G-map is invariant, hence every G-map from a compact G-space to a minimal G-space is onto.

A G-map that also is a homeomorphism is called an *isomorphism*, and two G-spaces are called *isomorphic* if there is an isomorphism between them.

The following is known, Ellis [5, 7.15], see also [3]:

1.2.1. Theorem. *For every group G there is a minimal compact G-space X such that*
(a) *for every minimal G-space Y there is a G-map from X onto Y; and*
(b) *every G map from a minimal G-space to X is an isomorphism.*

So up to isomorphism there is exactly one (compact) what we will call the *biggest minimal G-space.*[5] We will use BG to denote both this biggest minimal G-space and its underlying space. Our lofty goal is to characterize the space BG; we have no success, even in the case that interests us most, that of countable G, for example $G = \mathbb{Z}$.

Now what do we know about BG? It obviously has density at most $|G|$. Also, since BG can be constructed as a minimal invariant closed subset of βG, [5, 7.13], BG is a zero-dimensional F-space. Hence, if G is countable, then BG is extremally disconnected. (In fact, BG is known to always be extremally disconnected, [6]; we will give a proof in [4].)

For brevity call X an $\mathcal{S}$*-space* if X is a nonsingleton separable extremally disconnected compact Hausdorff space with the property that every nonempty clopen subset of X is homeomorphic to X. There are $2^{\mathfrak{c}}$ pairwise nonhomeomorphic $\mathcal{S}$-spaces, [3]. Well-known examples of $\mathcal{S}$-spaces are the absolutes of ${}^{\kappa}2$, with $\omega \leq \kappa \leq \mathfrak{c}$, where ${}^{\kappa}2$ denotes the product of κ factors 2.

We can show that if G is a countable group, then BG is an $\mathcal{S}$-space, and that BG is not homeomorphic to one of the quite exotic $\mathcal{S}$-spaces constructed in [3].

[4] Since G will always be clear from the context, we do not have to call this G-invariant.

[5] Ellis calls it a "universal minimal set". Since the G-map in Theorem 1.2.5(a) is not necessarily unique "universal" seems a misnomer.

This leads to the following:

1.2.2. Question. If G is a countable group, is BG homeomorphic to the absolute of ${}^{\kappa}2$ for some κ with $\omega \leqslant \kappa \leqslant \mathfrak{c}$?

An easy corollary to Theorem 1.1.3 is (see Section 10):

1.2.3. Theorem. *If G is an infinite Abelian group, then*
(a) $\pi(BG) > |G|$, *and*
(b) $|BG| = \exp^2|G|$, *indeed, $\beta|G|$ embeds into BG.*

Since the absolute of ${}^{\omega}2$ has countable π-weight, Theorem 1.2.7(a) implies a partial answer to Question 1.2.2:

1.2.4. Corollary. *If G is a countable Abelian group, then BG is not homeomorphic to the absolute of ${}^{\omega}2$.*

This is our original motivation for proving Lemma 1.1.2.

2. Preliminaries

2.1. Topology and sets

For any two sets A and B, ${}^{A}B$ denotes the set of functions $A \to B$. If B also is a space and/or a group, then ${}^{A}B$ also carries the product topology and/or product group structure; possible structure on A does not affect ${}^{A}B$.

If F is a set of functions $X \to Y$, then the *diagonal* of F is the function $\Delta F : X \to {}^{F}Y$ defined by

$$\Delta F(x)_f = f(x), \quad x \in X, f \in F.$$

Of course, ΔF is one-to-one iff F is point-separating, ΔF is continuous if F consists of continuous functions, and ΔF is a homomorphism if F consists of homomorphisms.

Recall from the introduction that $\mathbb{I}$, $\mathbb{N}$, $\mathbb{R}$ and 2 denote the unit interval $[0, 1]$, the positive integers, the reals and $\{0, 1\}$ respectively. Also recall that if S is a space, we say that a subspace A of a space X is *S-embedded* if every map $A \to S$ extends to a map $X \to S$. In Section 1 we stated:

2.1.1. Lemma. *Let X be any space, and let A be a subspace of X. If A is strongly zero-dimensional, in particular, if A is discrete, then*

$$\begin{array}{ccc} A \text{ is } \mathbb{N}\text{-embedded} & \Rightarrow & A \text{ is } 2\text{-embedded} \\ \Downarrow & & \Downarrow \\ A \text{ is } \mathbb{R}\text{-embedded} & \Rightarrow & A \text{ is } \mathbb{I}\text{-embedded.} \end{array}$$

Proof. It suffices to prove the downward implications.

If A is 2-embedded, then for every two disjoint zero sets Z_0 and Z_1 of A there is a map $f : A \to 2 \subseteq \mathbb{I}$ such that $(\forall i \in 2)[Z_i \subseteq f^{\leftarrow}\{i\}]$. Hence A is $\mathbb{I}$-embedded by [7, 1.17].

(We are admittedly lazy: One can also prove this implication directly, i.e., without using [7, 1.17].)

If A is $\mathbb{N}$-embedded, let f be any map $A \to \mathbb{R}$. For $n \in \mathbb{Z}$ choose a clopen set U_n in A such that $f^{\leftarrow}[n, n+1] \subseteq U_n \subseteq f^{\leftarrow}(n-1, n+2)$. One can define a function $m: A \to \mathbb{Z}$ by:

$$m(x) = \min\{n \in \mathbb{Z}: x \in U_n\};$$

of course m is a map. As A is $\mathbb{Z}$-embedded in X there is a map $\bar{m}: X \to \mathbb{Z}$ that extends m. Then $\mathscr{H} = \{\bar{m}^{\leftarrow}\{n\}; n \in \mathbb{Z}\}$ is a pairwise disjoint clopen cover of X such that $f \restriction H$ is bounded for every $H \in \mathscr{H}$. As A is $\mathbb{N}$-embedded in X, it is 2-embedded in X, hence $A \cap H$ is 2-embedded in H for each $H \in \mathscr{H}$, therefore $A \cap H$ is C^*-embedded ($\equiv \mathbb{I}$-embedded) in H, so we can choose a map $\phi_H : H \to \mathbb{R}$ that extends $f \restriction A \cap H$. Then $\bigcup\{\phi_H : H \in \mathscr{H}\}$ is a map $X \to \mathbb{R}$ that extends f. $\square$

We also remind the reader of the following well-known result.

2.1.2. Lemma. *If X is a countable regular space, then every closed subspace of X is $\mathbb{N}$-embedded in X.*

2.2. Abelian groups

Throughout this paper $\min\{\emptyset\} = \infty$; $(\forall n \in \mathbb{N})[n < \infty]$.

Throughout Section 2.2 G and H denote abstract Abelian groups.

The order of $x \in G$ is $\mathrm{o}(x) = \min\{n \in \mathbb{N}: n \cdot x = 0\}$.

For a subset A of G the subgroup of G generated by A is denoted by $\langle\!\langle A \rangle\!\rangle$.

G is called *divisible* if $(\forall m \in \mathbb{N})(\forall z \in G)(\exists y \in G)[m \cdot y = z]$.

The *circle group*, $\boldsymbol{T}$, is the subgroup $\{z \in \mathbb{R}^2: |z| = 1\}$ of the multiplicative complex plane $\mathbb{R}^2$. Obviously

$$\boldsymbol{T} \text{ is a divisible group such that } (\forall n \in \mathbb{N})(\exists x \in \boldsymbol{T})[\mathrm{o}(x) = n]. \qquad (*)$$

The *torsion group of* $\boldsymbol{T}$ is the subgroup $T = \{x \in \boldsymbol{T}: (\exists n \in \mathbb{N})[x^n = 1]\}$ of $\boldsymbol{T}$, or, equivalently, $T = \{e^{i\pi q}: q \in \mathbb{Q}\}$.

2.2.1. Lemma. *Let S be a subgroup of G and let h be a homomorphism $S \to H$ and let $x \in G$ and $y \in H$. Define $m \in \mathbb{N} \cup \{\infty\}$ by*

$$m = \min\{n \in \mathbb{N}: n \cdot x \in S\}.$$

(a) *h extends to a homomorphism $h': \langle\!\langle S \cup \{x\} \rangle\!\rangle \to H$ with $h'(x) = y$ iff $m = \infty$ or $m < \infty$ and $m \cdot y = h(m \cdot x)$; and*

(b) *if H is divisible and if $m \cdot y = h(m \cdot x)$, then h extends to a homomorphism $h'': G \to H$ such that $h''(x) = y$.*

2.2.2. Lemma. *G embeds algebraically in a power of $\boldsymbol{T}$.*

Proof. Let Γ denote the set of homomorphisms $G \to \boldsymbol{T}$. Γ is point-separating, i.e.,

$(\forall x \neq y \in G)(\exists h \in \Gamma)[h(x) \neq h(y)]$. To see this note that if $x \neq 0$, then there is a homomorphism $h: \langle\langle\{x\}\rangle\rangle \to T$ with $h(x) \neq 1$, because of $(*)$, and h extends to a homomorphism $G \to T$, because of $(*)$ and Lemma 2.1.1. Hence if b denotes the diagonal of Γ, i.e., b is the homomorphism $G \to {}^{\Gamma}T$ defined by

$$b(g)_h = h(g), \quad \text{for } g \in G, h \in \Gamma,$$

then b is an injection. □

A simple corollary that plays an important role in this paper is:

2.2.3. Lemma. *If H is an infinite subgroup of G, then there is an Abelian group D with $D \supseteq H$ and $|D| = |H|$ such that id_H extends to a homomorphism $G \to D$.*

Proof. Because of Lemma 2.2.1 it suffices to find a divisible group D with $D \supseteq H$ and $|D| = |H|$. By Lemma 2.2.2, H is a subgroup of a divisible group K, namely (up to isomorphism) ${}^{\kappa}T$ for some κ. For every infinite $A \subseteq K$ there is A^* with $A \subseteq A^*$ and $|A^*| = |A|$ such that $(\forall m \in \mathbb{N})(\forall a \in A)(\exists b \in A^*)[m \cdot b = a]$. Define $\langle D_n : n \in \omega \rangle$ by $D_0 = H$ and $D_{n+1} = \langle\langle (D_n)^* \rangle\rangle$. Then $D = \bigcup_{n \in \omega} D_n$ is as required. (See [8, proof of A.15] for another proof that D exists.) □

We also need the following result, in the proof of the case $\kappa = \omega$ of Lemma 1.1.2:

2.2.4. Lemma [8, A27]. *If G is finitely generated, then there are a finite K and $n \in \omega$ such that G is isomorphic to $K \times \mathbb{Z}^n$.*

3. The Bohr-compactification

In this section we briefly review the Bohr-compactification of a topological group.

For topological groups G and H we let $\mathrm{hom}(G, H)$ denote the set of all continuous homomorphisms $G \to H$. The *Bohr-compactification* of a topological group G is an essentially unique pair $\langle b_G, bG \rangle$, or $\langle b, bG \rangle$ for short if G is clear from the context, such that:

(a) bG is a compact group, $b \in \mathrm{hom}(G, bG)$, and $\mathrm{ran}(b)$ is dense in bG,

(b) for every compact group K and every $h \in \mathrm{hom}(G, K)$ there is $bh \in \mathrm{hom}(bG, K)$ such that $bh \circ b = h$.

This is analogous to the Čech–Stone compactification. However, b need not be one-to-one, see [8, pp. 348–351]. Even if b is one-to-one it need not be a homeomorphism. Indeed, it is known that b is a homeomorphism iff G is totally bounded.

(In particular, b never is a homeomorphism if G is a noncompact locally compact group. This can also be proved directly: A locally compact group never is a dense proper subgroup of a topological group, since locally compact dense subspaces are open, and since open subgroups are closed.)

If b is one-to-one let G^* denote G with that topology that makes $b: G \to bG$ an embedding. Of course, we will assume the function b actually is the identity. So G and G^* have the same underlying set, which is a subgroup of bG, and G^* also is a subspace of bG, but in general G is not. Condition (b) now becomes:

(b′) for every compact group K every member of $\hom(G, K)$ extends to a member of $\hom(bG, K)$.

If b is one-to-one, then the topology of G^* is the largest totally bounded group topology on G that makes id_G continuous as function $G \to G^*$. (Recall that a topological group G is called *totally bounded* if for every nonempty open U in G, there is a finite $F \subseteq G$ such that $F \cdot U = G$, were $F \cdot U$ denotes $\{fu : f \in F, u \in U\}$.)

Within the class of groups G such that b_G is one-to-one, bG is even more similar to βG: If G and H are any two such groups, then every continuous homomorphism $G \to H$ extends to a continuous homomorphism $bG \to bH$. To see this, consider any $h \in \hom(G, H)$. As id_H is continuous as function $H \to bH$, and $h = \mathrm{id}_H \circ h$, $h \in \hom(G, bH)$. Hence h extends to a member of $\hom(bG, bH)$.

For Abelian G one does not have to look at continuous homomorphisms from G to all compact groups K to define bG: First of all, one only has to look at Abelian K, of course. In fact it is sufficient to look only at $K = \mathbb{T}$, the circle group. (This is analogous to the fact that a compactification of a space X is βX iff X is $\mathbb{I}$-embedded in it.) This is so since if K is locally compact Abelian, then $\hom(K, \mathbb{T})$ is point-separating, hence if K is compact Abelian, then the diagonal of $\hom(K, \mathbb{T})$ embeds K topologically and algebraically into a power of $\mathbb{T}$. Therefore one can construct b and bG as follows: Let b be the diagonal map of $\hom(G, \mathbb{T})$, a map $G \to {}^G\mathbb{T}$, and let bG be $\overline{\mathrm{ran}(b)}$.

If $\hom(G, \mathbb{T})$ is point-separating, then the topology of G^* is the smallest topology that makes all functions in $\hom(G, \mathbb{T})$ continuous. So $\hom(G, \mathbb{T}) = \hom(G^*, \mathbb{T})$. Also, (b′) becomes:

(b″) every member of $\hom(G, \mathbb{T})$ extends to a member of $\hom(bG, \mathbb{T})$.

4. G^*

Throughout this section G and H denote discrete Abelian groups.

Recall from Section 3 that G^* denotes G with the smallest topology which makes every member of $\Gamma = \hom(G, \mathbb{T})$, the set of all homomorphisms $G \to \mathbb{T}$, continuous. In this section we collect relatively easy facts about the topology of G^*; for completeness sake we include all known results from the literature.

We begin with the known results.

4.1. Fact. *G^* satisfies the countable chain condition.*

Proof. This is so because G^* is a dense subgroup of the compact group bG □

4.2. Fact [2, p. 39]. *Every homomorphism $G \to H$ is continuous as function $G^* \to H^*$.*

4.3. Fact [2, 2.1]. *All subgroups of G are closed in G^*.*

Proof. If K is a subgroup of G, then $(G/K)^*$ is Tychonoff (since all H^* are Tynchonoff by Lemma 2.2.2, which says $\hom(H, \boldsymbol{T})$ is point-separating), hence T_1. Therefore K is closed by Fact 4.2. □

Note that an easy corollary is that $d(G^*) = |G|$.

4.4. Fact [2, p. 41]. *If K is a subgroup of G, then id_K is an embedding $K^* \to G^*$.*

Proof. id_K is continuous by Fact 4.2. id_K^{-1} is continuous since every member of $\hom(K, \boldsymbol{T})$ extends to a member of $\hom(G, \boldsymbol{T})$ because $\boldsymbol{T}$ is divisible. □

4.5. Fact [2, 2.5]. $(G \times H)^* \cong G^* \times H^*$.

Proof. Let i denote the identity function of the set $G \times H$.

To prove i is continuous as function $(G \times H)^* \to G^* \times H^*$ just note that the topology of $G^* \times H^*$ is the product topology of two totally bounded group topologies, hence is a totally bounded group topology on $G \times H$, and that the topology of $(G \times H)^*$ is the biggest totally bounded group topology on $G \times H$. Alternatively, note that $\pi_G \circ i$ and $\pi_H \circ i$ are continuous, where π_G and π_H are the projections $G \times H \to G$ and $G \times H \to H$, because of Fact 4.2.

To prove i is continuous as function $G^* \times H^* \to (G \times H)^*$ consider any neighborhood V of $\langle 1, 1\rangle$ in $(G \times H)^*$. There is a neighborhood U of $\langle 1, 1\rangle$ in $(G \times H)^*$ such that $U \cdot U \subseteq V$. By Fact 4.4 there are neighborhoods A of 1 in G^* and B of 1 in H^* such that

$$A \times \{1\} \subseteq U, \qquad \{1\} \times B \subseteq U.$$

Then $A \times B$ is a neighborhood of $\langle 1, 1\rangle$ in $G^* \times H^*$ such that $A \times B \subseteq V$ since for $a \in A$ and $b \in B$ one has

$$\langle a, b\rangle = \langle a, 1\rangle \cdot \langle 1, b\rangle \in U \cdot U \subseteq V. \qquad \square$$

We now come to new results, Theorems 4.6 and 4.8. $\mathbb{N}$-embedded was defined in Section 1.1.

4.6. Theorem. (a) *Every countable closed subspace of G^* is $\mathbb{N}$-embedded in G^*.*
(b) *Every countable subgroup of G^* is $\mathbb{N}$-embedded in G^*.*

Proof. (Of course (b) follows from (a), because of Fact 4.3; however, we will prove (a) from (b).)

In both parts of the proof we will use Lemma 2.1.2, i.e.,

$$\text{every closed subspace of a countable regular space is } \mathbb{N}\text{-embedded.} \qquad (*)$$

(b) Consider any countable subgroup K of G, and consider any map $f: K^* \to \mathbb{N}$. If G is countable, then by $(*)$ we could extend f to a map $G^* \to \mathbb{N}$. If G is not countable, we use Lemma 2.2.3: There is a countable group D with $D \supseteq K$ such that id_K extends to a map $r: G^* \to D^*$.

We may think of K^* as "K as subspace of G^*" and also as "K as subspace of D^*", by Fact 4.4. Thinking of K^* as "K as subspace of D^*" we see from $(*)$ that f can be extended to a map $\phi: D^* \to \mathbb{N}$. Then obviously $\phi \circ r$ is a map $G^* \to \mathbb{N}$ that extends f.

(a) Let A be a countable closed subspace of G. A is $\mathbb{N}$-embedded in the countable subgroup $\langle\langle A \rangle\rangle$ of G^*, by $(*)$. We know from (b) that $\langle\langle A \rangle\rangle$ is $\mathbb{N}$-embedded in G^*. Hence A is $\mathbb{N}$-embedded in G^*. $\square$

4.7. Corollary. *If G is infinite, then G^* is not pseudocompact,* [2, 2.2]. *In fact:*
(a) *there is a continuous function from G^* onto $\mathbb{N}$; and*
(b) *G^* is not a Baire space.*

(Recall that every pseudocompact space is a Baire space.)

Proof. (a) Let H be any countably infinite subgroup of G. Since H has no isolated points, H has an infinite closed discrete subset D. As H is a countable regular space, D is $\mathbb{N}$-embedded in H. Hence there is a map from H onto $\mathbb{N}$. This map extends to a map from G^* onto $\mathbb{N}$.

(b) The proof of Theorem 4.6 shows that there is a homomorphism from G onto a countable group, hence G has a subgroup H such that the collection $\mathscr{H}$ of cosets of H is countably infinite. H is closed in G^* by Fact 4.3, but is not open since G^* is totally bounded, and therefore H is nowhere dense. Hence $\mathscr{H}$ is a countably infinite cover of G^* by nowhere dense subsets. $\square$

Theorem 4.6 tells us that there are many maps $G^* \to 2$. Our next result tells us that the set of maps $G^* \to 2$ actually determines the topology of G^*; the following repeats Theorem 1.1.1.

4.8. Theorem. *G^* is zero-dimensional.*

Proof. We show that G^* has a subbase of clopen sets. To this end we consider any homomorphism $h: G \to T$ and any open neighborhood U of 1 in T, and prove there is a clopen neighborhood V of 1 in G^* such that V is a subset of the subbasic neighborhood $h^{\leftarrow}U$ of 1 in G^*. Before we embark on the proof we point out that the existence of V is trivial if G is a torsion group, for then $\mathrm{ran}(h)$ is a subset of the torsion group of T, which is countable, hence zero-dimensional.

T has nondegenerate subgroups A and B such that $A \cap B = \{1\}$ and $A \cdot B = T$. To see this let A be a nondegenerate divisible proper subgroup of T; as A is divisible, there is a homomorphism $\phi : T \to A$ that extends id_A, and $B = \phi^{\leftarrow}\{1\}$ is as required [8, A.8]. Alternatively, since T is an uncountable divisible group, it is a weak product of countable infinite groups [8, A.14].

Note that A and B are zero-dimensional since they are proper subgroups of T.

The function $m : A \times B \to T$, defined by

$$m(a, b) = a \cdot b, \quad \langle a, b \rangle \in A \times B$$

is continuous since it is the restriction of the multiplication of T. By our choice of A and B it is one-to-one. Hence m is a bijection.

Now consider the composition $m^{-1} \circ h$. In spite of m^{-1} being discontinuous (since it is a function from the connected space T onto the disconnected space $A \times B$), $m^{-1} \circ h$ is continuous: $A \times B$ is a subspace of $T \times T$, and if π denotes one of the two projections $T \times T \to T$, then $\pi \circ m^{-1} \circ h$ is a homomorphism $G \to T$, hence is it as continuous as function $G^* \to T$.

Recall that U is a neighborhood of 1 in T and that we need a clopen neighborhood V of 1 in G with $V \subseteq h^{\leftarrow} U$. Since m is continuous and since $A \times B$ is zero-dimensional, there is a clopen neighborhood W of $\langle 1, 1 \rangle$ in $A \times B$ such that $m^{\rightarrow} W \subseteq U$. Then $V = (m^{-1} \circ h)^{\leftarrow} W$, which clearly contains 1, is clopen, since $m^{-1} \circ h$ is continuous, and $(m^{-1} \circ h)^{\leftarrow} W \subseteq U$ since $(m^{-1} \circ h)^{\leftarrow} W = h^{\leftarrow}(m^{\rightarrow} W)$. □

Theorem 4.6 suggests several questions. We do not know whether "countable" is essential in Theorem 4.6(b). This leads to:

4.9. Question. Is every subgroup of G $\mathbb{N}$-embedded in G^*? 2-embedded in G^*? $\mathbb{R}$-embedded in G^*? $\mathbb{I}$-embedded in G^*?

Of course, the analogous question for Theorem 4.6(a) becomes:

4.10. Question. Is G^* strongly zero-dimensional? Normal?

While we do not know if G^* always is normal, we do know it is not paracompact in cases of interest:

4.11. Fact. *G^* is paracompact iff G^* is collectionwise Hausdorff iff G is countable.*

Proof. Of course, G^* is paracompact if G is countable since regular Lindelöf spaces are paracompact, and paracompact spaces are collectionwise Hausdorff.

G^* has the countable chain condition, by Fact 4.1, so if it is collectionwise Hausdorff, then it has no uncountable closed discrete subsets, hence then G is countable because of Theorem 1.1.3 and the remark just above Section 1.1. □

In the proof of Theorem 4.6 we used a homomorphism $r: G \to D$ that extends id_K. We did not really use the fact that r is a homomorphism, but only the consequence of Fact 4.2 that r is continuous as function $G^* \to D^*$. We ask whether one always can find such an r with $D = K$, i.e.,:

4.12. Question. Is every (countable) subgroup of G^* a retract of G^*?

A natural, because of Theorem 4.6(a), generalization of this question is:

4.13. Question. Is every countable closed subset of G^* a retract of G^*?

(The restriction to closed subsets that are countable is essential since by our main result G^* has an uncountable closed discrete subset (if G is uncountable), but G^* satisfies the countable chain condition by Fact 4.1.)

In Section 1 we pointed out that Theorem 1.1.3 implies that G^* has a closed discrete set of cardinality κ for every cadinal $\kappa \leqslant |G|$ that is not Ulam-measurable, and asked if the restriction on κ is essential, i.e.,:

4.14. Question. Does G^* have a closed discrete subset of cardinality $|G|$?

An obvious way to answer this question negatively would be to show that G^* always is realcompact. However, Alan Dow has pointed out:

4.15. Fact. *G^* is not realcompact if $|G|$ is Ulam-measurable.*

Proof. Because of Facts 4.3 and 4.4 assume without loss of generality that $|G|$ is the first Ulam-measurable cardinal. By Theorem 1.1.3, G^* has a relatively discrete $\mathbb{R}$-embedded subset D with $|D|$ the first Ulam-measurable cardinal. If G^* is realcompact, then $\bar{A} = \nu A$ for every $\mathbb{R}$-embedded subspace A of G. (νA denotes the realcompactification of A.) In particular, $\bar{D} = \nu D$. But $|\nu D| > |D|$. □

This leaves open the following:

4.16. Question. Is G^* realcompact if $|G|$ is not Ulam-measurable, in particular if $|G| = \omega_1$?

The answer to Question 4.14 will be an easy yes if the answer to the last question in this section is no:

4.17. Question. If G^* is infinite, does G^* have a relatively discrete subset that is not closed?

5. Simple simplifications

We first point out that the following weak version of Lemma 1.1.2 actually implies it:

5.1. Lemma. *Let G be an infinite Abelian group. Let $\mathscr{A}$ be a collection of at most $|G|$ subsets of G, each of cardinality exactly $|G|$. There is a subset T of G such that*:

(a) $(\forall A \in \mathscr{A})[|T \cap A| = |G|]$;

(b) *T is relatively discrete and $\mathbb{N}$-embedded in G^*; and*

(c) *T is $\mathbb{I}$-embedded in bG.*

Proof of Lemma 1.1.2 from Lemma 5.1. Let G be an infinite Abelian group, let κ be a cardinal with $\omega \leqslant \kappa \leqslant |G|$, and let $\mathscr{A}$ be a nonempty collection of at most κ subsets of G, each of cardinality exactly κ. Then $K = \langle\langle \bigcup \mathscr{A} \rangle\rangle$ is a subgroup of G with $|K| = \kappa$. By Lemma 2.2.3 there is an Abelian group A with $A \supseteq K$ and $|A| = |K|$ such that id_K extends to a homomorphism $r: G \to A$. Let D be as in Lemma 5.1 with A instead of G. We may assume $D \subseteq K$. Then (b) and (c) hold with A instead of G, and we claim they hold as stated: For each $S \subseteq K$, if S is $\mathbb{N}$-embedded in A^*, then S is $\mathbb{N}$-embedded in G^* since r is a map $G^* \to A^*$ that extends id_K, and similarly, if S is $\mathbb{I}$-embedded in bA, then S is $\mathbb{I}$-embedded in bG since r extends to a continuous function (even homomorphism) $br: bG \to bD$. $\square$

We next point out that in the countable case we get "$\mathbb{N}$-embedded" for free from $\mathbb{I}$-embedded and Lemma 2.1.2.

5.2. Lemma. *Assume every closed subspace of X is separable. Let $\mathscr{A}$ be an at most countable collection of countably infinite subsets of X. Every relatively discrete $\mathbb{I}$-embedded transversal for $\mathscr{A}$ has a subset that is closed in X and that still is a transversal.*

Proof. Assume D is not closed, and let $d: \omega \to \bar{D} \backslash D$ be such that $\overline{\mathrm{ran}(d)} = \bar{D} \backslash D$. Of course (as D is $\mathbb{I}$-embedded)

$$(\forall A \subseteq D)[\bar{A} \cap \overline{D \backslash A} = \emptyset]. \tag{$*$}$$

Since for every transversal P of $\mathscr{A}$ there is $Q \subseteq P$ such that both Q and $P \backslash Q$ are transversals, we see from $(*)$ that there is a sequence $\langle D_n : n \in \omega \rangle$ of transversals such that

$$D_0 \subseteq D, \qquad (\forall n \in \omega)[D_{n+1} \subseteq D_n], \qquad (\forall n \in \omega)[d_n \in \overline{D \backslash D_n}].$$

It is easy to find a transversal T of $\mathscr{A}$ such that

$$T \subseteq D, \quad \text{and} \quad (\forall n \in \omega)[T \backslash D_n \text{ is finite}].$$

We claim T is closed in X: For each $n \in \omega$ we have $d_n \in \overline{D \backslash D_n}$, hence $d_n \in \overline{D \backslash T}$ since $(D \backslash D_n) \backslash (D \backslash T) (= D \cap T \backslash D_n = T \backslash D_n)$ is finite and since $d_n \notin D \backslash D_n$. Hence

$\bar{D}\backslash D(=\overline{\operatorname{ran}(d)})\subseteq\overline{D\backslash T}$, therefore $\bar{D}\backslash T\subseteq\overline{D\backslash T}$. As $\overline{D\backslash T}\cap\bar{T}=\emptyset$, by (*), it follows that $\bar{T}=\bar{D}\backslash\overline{D\backslash T}\subseteq T$, as required. □

6. Solving equations in disjoint closed sets

It is not hard to prove that T has two disjoint closed subsets K and L such that for each $m\geqslant 3$

$$(\forall a\in T)(\exists x\in K)(\exists y\in L)[x^m=y^m=a]. \tag{1}$$

However, T does not have two disjoint closed subsets such that (1) holds for $m=2$, for otherwise $K\cup L=T$, since $(\forall x\in T)(\exists! y\in T)[x^2=y^2$ and $y\neq x]$, which contradicts the fact that T is connected. If we replace T by T^2, (1) is possible for all $m\geqslant 2$:

6.1. Lemma. *There are disjoint closed sets K and L in T^2 such that for each $m\geqslant 2$*

$$(\forall a\in T^2)(\exists x\in K^\circ)(\exists y\in L^\circ)[x^m=y^m=a]. \tag{2}$$

(*Here* $^\circ$ *denotes the interior operator of* T^2.)

Proof. Since T^2 is normal, it suffices to prove the lemma with (2) weakened to

$$(\forall a\in T^2)(\exists x\in K)(\exists y\in L)[x^m=y^m=a]. \tag{3}$$

In this proof we use $x\in\mathbb{I}$ as a name for $e^{i\pi x}\in T$. So, for example, we write mx for $(e^{i\pi x})^m$, and use $[a,b]$, with $0\leqslant a\leqslant b\leqslant 1$ to denote the "interval" $\{e^{i\pi x}: a\leqslant x\leqslant b\}$. Note that both 0 and 1, as elements of $\mathbb{I}$, are names for 1, as element of T.

Given two subsets K and L of T^2 and $m\geqslant 2$, how does one verify (3) holds? Well, for any $a\in T^2$, say $a=\langle p,q\rangle$ with $p,q\in\mathbb{I}$, there are $x,y\in[0,1/m]$ with $mx=p$ and $my=q$. From this we see that we prove (3) if we prove

$$\begin{gathered}(\forall x,y\in[0,1/m])(\exists s,t,u,v\in\omega)\\ [\langle x+s/m,y+t/m\rangle\in K \text{ and } \langle x+u/m,y+v/m\rangle\in L].\end{gathered} \tag{4}$$

We give K and L an easy to describe form: For suitable $p,q\in\mathbb{N}$ they are unions of nonoverlapping rectangles of size $p^{-1}\times q^{-1}$. Then K and L can be defined from partial matrices of 0's and 1's as we explain next.

If M is a $p\times q$ matrix, e.g.,

$$\left|\begin{array}{ccc|ccc}
\cdot & \cdot & 0 & 0 & \cdot & \cdot\\
1 & \cdot & 0 & 0 & \cdot & 1\\
1 & \cdot & \cdot & \cdot & \cdot & 1\\
\cdot & \cdot & \cdot & 1 & 1 & 1\\
\hline
\cdot & 0 & \cdot & 1 & 1 & 1\\
\cdot & 0 & \cdot & \cdot & 1 & \cdot\\
0 & 0 & 0 & \cdot & 1 & \cdot\\
0 & 0 & 0 & \cdot & \cdot & \cdot
\end{array}\right|$$

first divide T^2 into $p\times q$ "rectangles"

$$R_{jk}=[(j-1)/p, j/p]\times[(k-1)/q, k/q],\quad j=1,\dots,p \text{ and } k=1,\dots,q.$$

and then define K and L by

$$K=\bigcup\{R_{jk};\, M_{p-kj}=0\}\quad\text{and}\quad L=\bigcup\{R_{jk}:\, M_{p-kj}=1\}.$$

So K looks like the region of 0's of M and L looks like the region of 1's of M.

If for M we use the 6×8 matrix given above then, as is easy to see, $K\cap L=\emptyset$. Also, it satisfies (3) for $m=2$, since if we superimpose the four 3×4 submatrices that are indicated, then we get a 3×4 matrix such that every entry contains a 0 and a 1. M was found by trial and error. Further trial and error led to the following 12×12 matrix:

$$\left|\begin{array}{cccccc|cccccc}
\cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & 1 & 1 \\
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\cdot & \cdot & 0 & 0 & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 \\
\cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 \\
\cdot & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right|$$

Let us show that if K and L are defined from the matrix NEWM, then (4) holds for all $n\geqslant 2$:

As above, if we superimpose the four 6×6 submatrices that are indicated, then we get a 6×6 matrix such that every entry contains a 0 and a 1. Consequently (4) holds for $m=2$, and therefore it holds for all even $m\geqslant 2$.

Now consider any $m\geqslant 3$ that is odd, and consider any x, $y\in[0,1/m]$.

Step K: We find s and t. Since

$$[0,\tfrac{1}{3}]^2\setminus[0,\tfrac{1}{12})\times(\tfrac{1}{4},\tfrac{1}{3}]\subseteq K,$$

we can take $s=t=0$ except when $\langle x,y\rangle\in[0,\frac{1}{12})\times(\frac{1}{4},\frac{1}{3}]$ ($\subseteq R_{14}$). Then $m=3$ (since x, $y\in[0,1/m]$). Then we let $s=1$ and $t=0$; this works since $\langle\frac{1}{3},0\rangle+R_{14}=R_{54}$, a subset of K.

Step L: We show one can take $v=\frac{1}{2}(m-1)$, which is an integer since m is odd, and $u=m-1$. Since $\langle x,y\rangle\in[0,1/m]\times[0,1/m]$ and $m\geqslant 3$ we have

$$\left\langle x+\frac{u}{m}, y+\frac{v}{m}\right\rangle\in\left[1-\frac{1}{m},1\right]\times\left[\frac{1}{2}-\frac{1}{2m},\frac{1}{2}+\frac{1}{2m}\right]\subseteq\left[\frac{2}{3},1\right]\times\left[\frac{1}{3},\frac{2}{3}\right],$$

hence $\langle x+u/m, y+v/m\rangle\in L$. $\square$

The following analogue of Lemma 6.1 is a corollary to the lemma:

6.2. Lemma. *There is a countable (multiplicatively written) divisible subgroup Σ of ${}^{\omega}T$ and there is a countably infinite discrete collection $\mathscr{K}$ of closed subsets of Σ such that*

$$(\forall m \geqslant 2)(\forall a \in \Sigma)(\forall K \in \mathscr{K})(\exists x \in K)[x^m = a]. \tag{5}$$

Proof. Let T denote the torsion group of $\mathbb{T}$, and let S abbreviate the square T^2; we write S multiplicatively, with identity $\mathbb{1} = \langle 1, 1\rangle$. Rename the K and L of the lemma:

$$\text{NEW}K = \text{OLD}K \cap S, \qquad \text{NEW}L = \text{OLD}L \cap S;$$

note that $\mathbb{1} \notin L$. It should be clear that S is countable and that

$$S \text{ is divisible and } (\forall a \in T^2)(\exists x \in K)(\exists y \in L)[x^n = y^n = a]. \tag{6}$$

Now let Σ denote the sum of ω summands S, i.e.,

$$\Sigma = \{x \in {}^{\omega}S \colon x_n = \mathbb{1} \text{ for all but finitely many } n \in \omega\}.$$

It is clear that Σ is countable and divisible. For $n \in \omega$ define $K_n \subseteq \Sigma$ by

$$K_n = \{x \in \Sigma \colon (\forall k \in n)[x_k \in L] \text{ and } x_n \in K\}.$$

We will prove Σ and $\mathscr{K} = \langle K_n : n \in \omega\rangle$ are as required.

Clearly $\mathscr{K}$ is a collection of sets that are closed in Σ. To see $\mathscr{K}$ is a discrete indexed collection in Σ note that if x is any point of ${}^{\omega}S$ such that every neighborhood of x meets infinitely many members of $\mathscr{K}$, then $(\forall n \in \omega)[x_n \in L]$, so that $x \notin \Sigma$ because $\mathbb{1} \notin L$.

To prove (5) consider any $a \in \Sigma$ and any $p \geqslant 2$ and any $n \in \omega$. Let $m > n$ be such that $(\forall k > m)[a_k = \mathbb{1}]$. There is $x \in {}^{\omega}S$ such that

$$(\forall k \in n)[x_k \in L], \quad x_n \in K, \quad (\forall k \leqslant m)[x_k^p = a_k], \quad (\forall k > m)[x_k = 1]. \qquad \square$$

7. The case $\kappa > \omega$ of Lemma 5.1

We must prove:

> Let G be an Abelian group of cardinality $\gamma > \omega$. Every collection $\mathscr{F} \subseteq [G]^{\gamma}$ with $|\mathscr{F}| \leqslant \gamma$ has a transversal that is relatively discrete in G^*, is $\mathbb{N}$-embedded in G^* and is $\mathbb{I}$-embedded in bG. (*)

Proof. It is easy to find $D \subseteq G$ and a well-order $<$ of D (in type γ) such that

(1) $(\forall F \in \mathscr{F})[|D \cap F| = \gamma]$;
(2) $(\forall x \in D)[x \notin \langle\!\langle \{y \in D \colon y < x\}\rangle\!\rangle]$; and
(3) $\langle\!\langle D \rangle\!\rangle = G$.

(Condition (3) is not really essential, but has a minor technical advantage.)

Clearly D is a transversal.

To prove D is relatively discrete in bG and is $\mathbb{I}$-embedded in bG we consider any $S \subseteq D$, and prove $\mathrm{cl}_{bG} S \cap \mathrm{cl}_{bG}(S \setminus D) = \emptyset$. Let K and L be the disjoint closed subsets of T^2 given by Lemma 7.1. Because of Lemma 2.2.1 there is an easy transfinite construction of a homomorphism h from $\langle\langle D\rangle\rangle = G$ into T^2 such that $(\forall x \in S)[h(x) \in K]$ and $(\forall x \in D\backslash S)[h(x) \in L]$. As h extends to a homomorphism $bG \to T^2$, it follows that $\mathrm{cl}_{bG} S \cap \mathrm{cl}_{bG}(S\backslash D) = \emptyset$.

To prove D is relatively discrete in $G^\#$ and is $\mathbb{N}$-embedded in $G^\#$ we prove

$$\text{every function } D \to \mathbb{N} \text{ extends to a map } G^\# \to \mathbb{N}. \qquad (**)$$

Let Σ and $\mathscr{K}$ be as in Lemma 6.2. Think of $\mathscr{K}$ as a discrete space, so we may replace $\mathbb{N}$ by $\mathscr{K}$ in (**). Since Σ is a countable regular space, $\bigcup \mathscr{K}$ is $\mathbb{N}$-embedded in Σ, by Lemma 2.1.2, hence there is a map $k : \Sigma \to \mathscr{K}$ such that $(\forall K \in \mathscr{K})[K \subseteq k^{\leftarrow}\{K\}]$.

Now consider any function $f : D \to \mathscr{K}$. As in the proof above one can construct a homomorphism $h : G \to \Sigma$ such that $(\forall x \in D)[h(x) \in f(x)]$; obviously $k \circ h$ extends f. As h is continuous as function $G^\# \to \Sigma$, it follows that $k \circ h$ is continuous as function $G^\# \to \mathscr{K}$. $\square$

8. The case $\kappa = \omega$ of Lemma 5.1

We must prove:

> Let $\mathscr{A}$ be an at most countable collection of countable infinite subsets of G. Then $\mathscr{A}$ has a countable relatively discrete transversal that is $\mathbb{I}$-embeded in bG. (*)

(Actually, we only have to prove this for countable G. However, the condition that G be countable plays no rôle in the proof of "$\mathbb{I}$-embedded". Recall that "$\mathbb{N}$-embedded" follows from Lemma 5.2 in the case that G is countable.)

Proof. Let Γ denote the subspace of ${}^G(T^2)$ consisting of all homomorphisms $G \to T^2$. As

$$\Gamma = \{h \in {}^G(T^2) : (\forall x, y \in G)[h(x+y) = h(x) + h(y)]\},$$

Γ is closed in ${}^G(T^2)$, hence Γ is compact.

By Lemma 6.1 there are disjoint closed sets K_0 and K_1 in T^2 such that

$$(\forall m \geqslant 2)(\forall a \in T^2)(\exists x \in \operatorname{int} K_0)(\exists y \in \operatorname{int} K_1)[x^m = y^m = a]. \qquad (**)$$

Claim 1. *For each infinite $A \subseteq G$, for every finite family $\mathscr{U}$ of nonempty open subsets of Γ there is an $a \in A$ such that*

$$(\forall U \in \mathscr{U})(\forall i \in 2)(\exists \text{ nonempty open } V \text{ in } \Gamma)$$
$$[\bar{V} \subseteq U \text{ and } (\forall h \in V)[h(a) \in K_i]].$$

Proof of statement (*) from Claim 1. We will use the tree ${}^{<\omega}2$, i.e., $\bigcup_{n \in \omega} {}^n2$. Recall that for $s \in {}^{<\omega}2$ and $i \in 2$ one uses $s\hat{\ }i$ and to denote the concatenation of s and i, i.e., $s\hat{\ }i = s \cup \{\langle \mathrm{dom}(s), i\rangle\}$.

Enumerate $\mathscr{A}$ as $\langle A_n : n \in \omega \rangle$ such that each $A \in \mathscr{A}$ occurs at infinitely many places in this sequence. Using recursion we can construct a choice function $\alpha \in \prod_{n\in\omega} A_n$ and a function U from ${}^{<\omega}2$ to the collection of nonempty open subsets of Γ such that for every $n \in \omega$ we have:

(1) $(\forall t \in {}^n2)(\forall i \in 2)(\forall h \in U(t^\frown i))[h(\alpha_n) \in K_i]$; and

(2) $(\forall t \in {}^n2)(\forall i \in 2)[\mathrm{cl}_\Gamma\, U(t^\frown i) \subseteq U(t)]$.

The construction is straightforward: Let $U(\emptyset) = \Gamma$. For $n \in \omega$, if $\mathscr{U} = \langle U(t) : t \in {}^n2 \rangle$ is known, use Claim 1 to find $\langle U(s) : s \in {}^{n+1}2 \rangle = \langle U(t^\frown i) : t \in {}^n2, i \in 2 \rangle$.

Clearly $D = \mathrm{ran}(\alpha)$ is a transversal of $\mathscr{A}$. to prove D is relatively discrete in G^* and is $\mathbb{1}$-embedded in bG it suffices to prove that every two disjoint subsets of D have disjoint closures in bG. So consider any $s \in {}^\omega 2$, we will prove $D_0 = \{\alpha_n : s_n = 0\}$ and $D_1 = \{\alpha_n : s_n = 1\}$ have disjoint closures in bG. To this end it suffices to find a homomorphism $h: G \to T^2$ such that $(\forall i \in 2)[h^{\to} D_i \subseteq K_i]$, or, equivalently, such that

(3) $(\forall n \in \omega)[h(\alpha_n) \in K(s_n)]$.

Since Γ is compact, there is an h in $\bigcap_{n\in\omega} U(s \upharpoonright n)$, because of (2), and this h satisfies (3) because of (1).

We introduce some notation. Γ will be a multiplicatively written group, with identity $\mathbb{1}$, so $\mathbb{1}(g) = 1$ for $g \in G$. Also, for $\varepsilon > 0$ define

$$I(\varepsilon) = \{e^{i\pi x} : x \in (-\varepsilon, \varepsilon)\},$$

a basic neighborhood of 1 in T, and for finite $F \subseteq G$ and for $\varepsilon > 0$ define

$$B(F, \varepsilon) = \{h \in \Gamma : (\forall x \in F)[h(x) \in I(\varepsilon)^2]\},$$

a basic neighborhood of $\mathbb{1}$ in Γ. Note that each $p \in \Gamma$ has arbitrarily small neighborhoods of the form $p \cdot B(F, \varepsilon)$, with $F \subseteq G$ finite and with $\varepsilon > 0$.

Claim 1 is rather unpleasant. We will derive it from the following:

Claim 2. *Let P be a finite subset of G. Let M be an infinite subset of $\mathbb{Z}$, let $\varepsilon > 0$, and let*

$$\langle I_{pmi} : p \in P, m \in M, i \in 2 \rangle$$

be an indexed collection of nonempty subsets of T^2. There is an $m \in M$ and an $s: P \times 2 \to T$ such that

$$(\forall p \in P)(\forall i \in 2)(\exists s \in T^2)[s \in I(\varepsilon)^2, s^m \in I_{pmi}].$$

This may not look much better, so let us first show Claim 2 is easy:

Proof of Claim 2. For each $m \in M$ the "square" $I_m = \{s^m : s \in I(\varepsilon)^2\}$ has sides of length $2\varepsilon \cdot |m|$, since $I(\varepsilon)$ has length 2ε, hence since M is an infinite subset of $\mathbb{Z}$ there is $m \in M$ such that $I_m = T$, hence such that $I_{pmi} \subseteq I_m$ for $p \in P$. Hence for each $p \in P$ and $i \in 2$ there is $s \in I(\varepsilon)$ such that $s^m \in I_{pmi}$.

Claim 1 easily follows from a slightly weaker version:

Claim 1′. *For each infinite $A \subseteq G$, for every finite family $\mathscr{U}$ of nonempty open subsets of Γ there is an $a \in A$ such that*

$$(\forall U \in \mathscr{U})(\forall i \in 2)(\exists h \in \Gamma)[h \in U, h(a) \in \mathrm{int}\, K_i].$$

Proof of Claim 1 from Claim 1′. Given $U \in \mathcal{U}$, $i \in 2$ and $h \in U$ with $h(a) \in K_i$ let V be a sufficiently small neighborhood of h.

Proof of Claim 1′ from Claim 2. Consider any infinite $A \subseteq G$ and any finite family $\mathcal{U}$ of nonempty open subsets of Γ. We may assume there are finite $F \subseteq G$, $P \subseteq \Gamma$ and $\varepsilon > 0$ such that

$$\mathcal{U} = \{p \cdot B(F, \varepsilon) : p \in P\}.$$

Our objective is to find $a \in A$ such that

$$(\forall p \in P)(\forall i \in 2)(\exists h \in p \cdot B(F, \varepsilon))[h(a) \in \operatorname{int} K_i], \tag{a}$$

or, equivalently, such that

$$(\forall p \in P)(\forall i \in 2)(\exists h \in B(F, \varepsilon))[h(a) \in p(a)^{-1} \cdot \operatorname{int} K_i], \tag{b}$$

Case 1: $A \not\subseteq \langle\!\langle F \rangle\!\rangle$. Let $a \in A \backslash \langle\!\langle F \rangle\!\rangle$, and put $m = \inf\{k \in \mathbb{N} : a \in \langle\!\langle F \rangle\!\rangle\}$; note that $m \geqslant 2$ since $a \notin \langle\!\langle F \rangle\!\rangle$. Fix $p \in P$ and $i \in 2$. Choose $x \in \operatorname{int} K_i$ such that if $m < \infty$, then $x^m = p(m \cdot a)$; by Lemma 2.2.1 there is a homomorphism $h : G \to T^2$ which extends $p \restriction \langle\!\langle F \rangle\!\rangle$ such that $h(a) = x$. Clearly $h \in p \cdot B(F, \varepsilon)$ and $h(a) \in \operatorname{int} K_i$.

Case 2: $A \subseteq \langle\!\langle F \rangle\!\rangle$. Since $\langle\!\langle F \rangle\!\rangle$ is an infinite finitely generated group, it follows from Lemma 2.2.4 that $\langle\!\langle F \rangle\!\rangle$ is isomorphic to $K \times \mathbb{Z}^n$ for some finite group K and for some $n \in \mathbb{N}$. We assume $\langle\!\langle F \rangle\!\rangle$ is $K \times \mathbb{Z}^n$, and let a point x of $\langle\!\langle F \rangle\!\rangle$ have coordinates $x_0, \ldots, x_n$ with $x_0 \in K$ and with $x_i \in \mathbb{Z}$ for $i \in \{1, \ldots, n\}$. Since A is infinite but K and n are finite, there is $q \in \{1, \ldots, n\}$ such that $M = \{a_q : a \in A\}$ is infinite. Choose $\alpha : M \to A$ such that

$$(\forall m \in M)[\alpha(m)_q = m].$$

Let

For each $s \in T$ there is a homomorphism $h_s : G \to T$ such that

$$(\forall x \in \langle\!\langle F \rangle\!\rangle)[h_s(x) = s^{x(q)}].$$

Then

$$\text{for each } m \in M \text{ we have } h_s(\alpha_m) = s^m. \tag{c}$$

Let μ abbreviate $\max_{x \in F} |x_q|$. Of course, $\mu > 0$. We claim that

$$(\forall s \in I(\varepsilon/\mu))[h_s \in B(F, \varepsilon)]. \tag{d}$$

Indeed, for $x \in F$ and $s \in I(\varepsilon/\mu)$

$$h_s(x) = s^{x(q)} \in I(|x_q| \cdot \varepsilon/\mu) \subseteq I(\mu \cdot \varepsilon/\mu) = I(\varepsilon).$$

Now apply Claim 2 with $I_{pmi} = p(\alpha_m)^{-1} \cdot \operatorname{int} K_i$ for $p \in P$, $m \in M$, $i \in 2$, to find $m \in M$ and $s : P \times 2 \to T^2$ such that

$$(\forall p \in P)(\forall i \in 2)[s(p, i) \in I(\varepsilon/\mu)^2];$$

$$(\forall p \in P)(\forall i \in 2)[s(p, i)^m \in p(\alpha_m)^{-1} \cdot \operatorname{int} K_i].$$

If $b_{pi} = h_{s(p,i)}$, then from (c) and (d)

$$(\forall p \in P)(\forall i \in 2)[b_{pi}(\alpha_m) = s(p, i)^m \in p(\alpha_m)^{-1} \cdot \operatorname{int} K_i, b_{pi} \in B(F, \varepsilon)]. \qquad \square$$

9. The Bohr-compactification of $\mathbb{R}$

We will prove a theorem about $b\mathbb{R}$ which is the $\kappa = \omega$ analogue of Lemma 1.1.2:

9.1. Theorem. *For an at most countable collection $\mathscr{A}$ of countable subsets of $\mathbb{R}$ the following are equivalent:*

(a) $(\forall A \in \mathscr{A})[\mathrm{cl}_{\mathbb{R}} A$ *is not compact*];

(b) $\mathscr{A}$ *has a transversal that is relatively discrete and* $\mathbb{I}$*-embedded in* $\mathbb{R}^*$;

(c) $\mathscr{A}$ *has a transversal that is closed discrete and* $\mathbb{R}$*-embedded in* $\mathbb{R}^*$, *and that is* $\mathbb{I}$*-embedded in* $b\mathbb{R}$.

We prove this from a lemma that is somewhat stronger than we really need. For an interval I in T or $\mathbb{R}$ we use $\lambda(I)$ to denote its length.

9.2. Lemma. *Let $\mathscr{A}$ be an at most countable collection of unbounded subsets of $\mathbb{R}$ and let σ be a sequence of positive real numbers. There is an injection $a : \omega \to \mathbb{R}$ such that*

(a) $(\forall A \in \mathscr{A})[\{n \in \omega : a_n \in A\}$ *is infinite*]; *and*

(b) *for every sequence $\langle I_k : k \in \omega \rangle$ of closed intervals in T such that* $(\forall k \in \omega)$ $[\lambda(I_k) = \sigma_k]$ *there is a homomorphism $h : \mathbb{R} \to T$ such that $h(a_k) \in I_k$ for $k \in \omega$.*

Proof. For $t \in \mathbb{R}$ define the homomorphism $e_t : \mathbb{R} \to T$ by

$$e_t : x \mapsto \exp(itx), \quad x \in \mathbb{R}.$$

It is known that every continuous homomorphism $\mathbb{R} \to T$ has the form e_t for some $t \in \mathbb{R}$.

In what follows we use the fact that for $x \in \mathbb{R}$ and for intervals $K \subseteq \mathbb{R}$:

$$\{e_t(x) : t \in K\} = e_x^{\to} K, \qquad \lambda(xK) = |x| \cdot \lambda(K). \tag{1}$$

Given $a : \omega \to \mathbb{R}$ and a sequence $\langle I_k : k \in \omega \rangle$ of closed intervals in T such that $(\forall k \in \omega)[\lambda(I_k) \geqslant \sigma_k]$ how do we find $t \in \mathbb{R}$ such that $e_t(a_k) \in I_k$ for $k \in \omega$? By finding a nested sequence $\langle J_k : k \in \omega \rangle$ of bounded closed intervals in $\mathbb{R}$ such that $(\forall k \in \omega)$ $(\forall t \in J_k)[e_t(a_k) \in I_k]$. Of course, in order to have room to play we take the J_k as big as possible, hence for the each $k \in \omega$ we will have:

$$\{e_t(a_k) : t \in J_k\} = I_k. \tag{2}$$

From this and (1) we see that

$$(\forall k \in \omega)[|a_k| \cdot \lambda(J_k) = \lambda(I_k)(=\sigma_k)]. \tag{3}$$

During a recursive construction, if we have J_k, how do we know we can find J_{k+1} such that (2) holds for $k+1$? Let $\mathscr{R}$ and $\mathscr{T}$ denote the collection of closed intervals in $\mathbb{R}$ and in $\mathbb{T}$, respectively, and draw a picture to see that

$$(\forall J \in \mathscr{R})(\forall x \in \mathbb{R})(\forall \delta \in (0, 2\pi])[(\forall I \in \mathscr{T})$$
$$[\lambda(I) = \delta \Rightarrow (\exists J' \in \mathscr{R})[J' \subseteq J, \{e_t(x): t \in J'\} = I]$$
$$\text{if (and only if) } \lambda(\{e_t(x): t \in J\}) \geq 2\pi + \delta]].$$

Hence we must have $\lambda(\{e_t(a_{k+1}): t \in J_k\}) \geq 2\pi + \sigma_{n+1}$. From this and (1) we see that we can construct $\langle J_k : k \in \omega\rangle$ provided for each $k \in \omega$ we have $|a_{k+1}| \cdot \lambda(J_k) \geq 2\pi + \sigma_{k+1}$. Substitution into (3) yields

$$|a_{k+1}| \geq (2\pi + \sigma_{k+1})/\lambda(J_k) = |a_k| \cdot (2\pi + \sigma_{k+1})/\sigma_k.$$

Since $\mathscr{A}$ consists of at most countably many infinite unbounded subsets of $\mathbb{R}$ it follows that we can find $a : \omega \to \mathbb{R}$ as required. □

We now proceed to the proof of Theorem 9.1.

Proof of Theorem 9.1. (a)$\Rightarrow$(c). We first find a relatively discrete transversal of $\mathscr{A}$ that is $\mathbb{I}$-embedded in $b\mathbb{R}$:

Let F and G be two disjoint closed intervals in $\mathbb{T}$ of equal length. By Lemma 9.2 there is a transversal D of $\mathscr{A}$ such that for every function $\phi : D \to \{F, G\}$ there is $h \in \text{hom}(\mathbb{R}, \mathbb{T})$ such that $\phi^{\leftarrow}\{F\} \subseteq h^{\leftarrow}F$ and $\phi^{\leftarrow}\{G\} \supseteq h^{\leftarrow}G$. As F and G are disjoint closed subsets of $\mathbb{T}$ this shows D is relatively discrete and $\mathbb{I}$-embedded in $b\mathbb{R}$.

As $\mathbb{R}$ is hereditarily separable, so is its continuous image $\mathbb{R}^*$. It now follows from Lemma 5.2 $\mathscr{A}$ has a closed transversal D' with $D' \subseteq D$.

Finally, since $\mathbb{R}$ is Lindelöf, so is its continuous image $\mathbb{R}^*$. Hence $\mathbb{R}^*$ is normal. Therefore the closed discrete set D' of $\mathbb{R}^*$ is $\mathbb{R}$-embedded in $\mathbb{R}^*$.

(c)$\Rightarrow$(b). This is trivial.

(b)$\Rightarrow$(a). We prove the contrapositive: Let A be an infinite bounded element of $\mathscr{A}$, and let D be a relatively discrete transversal of $\mathscr{A}$. Since D is relatively discrete in $\mathbb{R}^*$ and since $\text{id}_{\mathbb{R}}$ is a continuous bijection $\mathbb{R} \to \mathbb{R}^*$, to prove D is not $\mathbb{I}$-embedded in $\mathbb{R}^*$, it suffices to prove $B = D \cap A$ is not $\mathbb{I}$-embedded in $\mathbb{R}$: B is bounded, hence $\text{cl}_{\mathbb{R}}B$ is compact metrizable, so B, being infinite, is not $\mathbb{I}$-embedded in $\text{cl}_{\mathbb{R}}B$, hence not in $\mathbb{R}$. □

10. *BG* and *bG* and *G**

In this section we prove Theorem 1.2.3 from Lemma 1.1.2.

Since G^* is dense in the topological group bG, so is Gx for every $x \in bG$. Hence bG is a compact minimal G-space. Therefore we prove Theorem 1.2.3 if we prove:

10.1. Fact. *If K is any minimal compact G-space which admits a G-map onto bG, then*
(a) $\pi(K) > |G|$; *and*
(b) $\beta|G|$ *embeds in* K.

We prepare for the proof. Call a G-space X *totally bounded* if for every nonempty open $U \subseteq X$ there is a finite $F \subseteq G$ such that $X = F \cdot U$. The following known result, [5, 2.5], is easy to prove:

10.2. Lemma. *Let X be a compact G-space, consider any $x \in X$. X is a minimal G-space iff $\overline{Gx} = X$ and Gx is totally bounded.*

Proof of Fact 10.1. Let f be a G-map from K onto bG, and consider any $p \in K$ with $f(p) = \mathbb{0}$, the identity of G. Since K is a minimal G-space, Gp is dense in K.

Observe that $f \restriction Gp$ is continuous bijection from Gp onto G^*. Hence from Lemma 1.1.2 we see that the following analogue of Lemma 1.1.2 holds:

> Let κ be a cardinal with $\omega \leqslant \kappa \leqslant |G|$, let $\mathscr{A}$ be a collection of at most κ subsets of Gp, each of cardinality exactly κ. There is a subset D of Gp such that:
> (A) $(\forall A \in \mathscr{A})[|D \cap A| = \kappa]$;
> (B) D is relatively discrete and $\mathbb{N}$-embedded in Gp; and
> (C) D is $\mathbb{I}$-embedded in K. (*)

(a) $\pi(K) = \pi(Gp)$ since $\pi(X) = \pi(Y)$ whenever X is a dense subspace of a (regular) space Y. Therefore it suffices to prove $\pi(Gp) > |G|$.

From Lemma 10.2 we know that each nonempty open subset of Gp has cardinality γ, in particular, Gp has no isolated points.

Now consider any collection $\mathscr{A}$ of at most $|G|$ nonempty open subsets of Gp. Since all members of $\mathscr{A}$ have cardinality $|G|$, it follows from (*) that there is a relatively discrete $D \subseteq Gp$ such that $(\forall A \in \mathscr{A})[A \cap D \neq \emptyset]$. But Gp has no isolated points, hence D is not dense in Gp. Therefore $Gp \setminus \bar{D}$ is a nonempty open subset of Gp such that there is no $A \in \mathscr{A}$ with $A \subseteq Gp \setminus \bar{D}$.

(b) Because of (*) Gp has a relatively discrete subset D with $|D| = |G|$ that is $\mathbb{I}$-embedded in K. Of course $\bar{D}$ is a homeomorph of $\beta|G|$. □

References

[1] W.W. Comfort and K.A. Ross, Topologies induced by groups of characters, Fund. Math. 55 (1964) 283–291.
[2] W.W. Comfort and V. Saks, Countably compact groups and finest totally bounded topologies, Pacific J. Math. 49 (1973) 33–44.
[3] E.K. van Douwen, Barely separable extremally disconnected compacts, in preparation.
[4] E.K. van Douwen, Maximal topologies, in preparation.
[5] R. Ellis, Lectures on Topological Dynamics (Benjamin, New york, 1969).
[6] R. Ellis, personal communication, 1986.
[7] L. Gillman and M. Jerison, Rings of Continuous Functions (Van Nostrand Reinhold, New York, 1960).
[8] E. Hewitt and K.A. Ross, Abstract harmonic analysis I, Grundlehren 115 (Springer, Berlin, 1963).
[9] V. Kuz'minov, On a hypothesis of P.S. Alexandroff in the theory of topological groups, Dokl. Akad. Nauk SSSR 125 (1959) 727–729.

[10] W. Rudin, Fourier analysis on groups, Interscience, 1962.

[11] V. Uspenskii, Why topological groups are dyadic, in: Z. Frolík, ed., General Topology and Its Relations of Modern Analysis and Algebra, Proceedings 6th Prague Topology Symposium (Heldermann, 1986) 601-610.

Notes by the referee

(1) Theorem 1.1.3(d) is orginally due to G.A. Reid (Math. Z. 102 (1967) 227-235). For a short and quite simple proof, see P. Flor, Math. Scand. 23 (1968) 169-170.

(2) That BG is extremally disconnected is quite well known, although it is difficult to find a reference. The following simple proof was communicated to the referee by P. Simon, who attributed it to B. Balcar. By construction, BG is a minimal right ideal in the semigroup βG, [5, 7.13]. Note that BG contains an idempotent u and that $p \to up : \beta G \to BG$ is a retraction [5, 3.5]. Being a retract of the extremally disconnected space βG, BG is extremally disconnected as well.

(3) That $(G \times H)^* \equiv G^* \times H^*$ (Fact 4.5) is an obvious consequence of the fact that $b(G \times H) = bG \times bH$ (canonically), an "equality" that holds for arbitrary products of arbitrary topological groups (even semigroups). For the case under consideration (a finite product of Abelian groups) this equality follows from a result in K. de Leeuw and I. Glicksberg, Acta Math. 105 (1961) 63-97. See also P. Holm, Math. Annalen 156 (1964) 34-46 (the proof presented there works only for Abelian groups).